

Existence and stability of a blow-up solution with a new prescribed behavior for a heat equation with a critical nonlinear gradient term

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Abstract

We consider the semilinear heat equation, to which we add a nonlinear gradient term, with a critical power. We construct a solution which blows up in finite time. We also give a sharp description of its blow-up profile. The proof relies on the reduction of the problem to a finite dimensional one, and uses the index theory to conclude. Thanks to the interpretation of the parameters of the finite-dimensional problem in terms of the blow-up time and point, we also show the stability of the constructed solution with respect to initial data. This note presents the results and the main arguments. For the details, we refer to our paper [32].

Existence et stabilité d'une solution explosive avec un nouveau comportement prescrit pour une équation de la chaleur avec un terme en gradient non linéaire et critique. On considère l'équation semi-linéaire de la chaleur, à laquelle on rajoute un terme non linéaire en gradient, avec puissance critique. On montre l'existence d'une solution explosant en temps fini uniquement à l'origine, et on en donne le profil à l'explosion. Notre méthode s'appuie sur la réduction du problème en dimension finie, puis la solution de ce problème grâce à un argument topologique. Grâce à l'interprétation des paramètres du problème de dimension finie en terme du choix du temps et du point d'explosion, on obtient la stabilité de la solution construite par rapport aux données initiales. Cette note présente les résultats et les arguments de la preuve. Pour les détails, voir notre papier [32].

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

In our paper [32], we consider the following nonlinear heat equation:

$$\begin{aligned}\partial_t u &= \Delta u + \mu |\nabla u|^q + |u|^{p-1} u, \\ u(\cdot, 0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N),\end{aligned}\tag{1.1}$$

where $u = u(x, t) \in \mathbb{R}$, $t \in [0, T)$, $x \in \mathbb{R}^N$, and the parameters μ , p and q are such that

$$\mu > 0, \quad p > 3, \quad q = q_c \equiv \frac{2p}{p+1}.\tag{1.2}$$

This equation was first introduced by Chipot and Weissler [4] in 1989 with $\mu < 0$ (and $q > 1$) as a model to understand whether blow-up may be prevented by the addition of the negative gradient term. Later in 1996, Souplet suggested a population dynamics interpretation for the equation in [25]. Many authors dealt with the mathematical analysis of this equation, both for large time dynamics and finite time blow-up, and also for the elliptic version (see [4, 5, 26, 27, 24, 23, 28, 29, 30] and references therein).

Equation (1.1) enjoys two limiting cases:

- when $\mu = 0$, we have the well-known *semilinear heat equation*:

$$\partial_t u = \Delta u + |u|^{p-1} u;$$

- when $\mu \rightarrow \infty$, we recover (after appropriate rescaling) the *diffusive Hamilton-Jacobi equation*:

$$\partial_t u = \Delta u + |\nabla u|^q.$$

The value $q = 2p/(p+1)$ is critical, since for $\mu \neq 0$, it is the only value for which equation (1.1) is invariant under the transformation: $u_\lambda(t, x) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x)$, as for the equation without the gradient term, that is when $\mu = 0$. Moreover, we know from the literature that both the blow-up and the large-time behaviors depend on the position of q with respect to $\frac{2p}{p+1}$ (see the above-mentioned literature).

Equation (1.1) is wellposed in $W^{1,\infty}(\mathbb{R}^N)$ thanks to a fixed point argument (see also [1], [30] and [31]). In our paper [32], we focus on the study of blow-up for that equation.

When $\mu = 0$, there is a huge literature about the subject, and no bibliography can be exhaustive. Let us focus on the existence a stable solution $u(x, t)$ which blows up in finite time $T > 0$ only at the origin and satisfies

$$(T - t)^{1/(p-1)} u(z \sqrt{(T - t)|\log(T - t)|}, t) \sim f_0(z) \text{ as } t \rightarrow T,\tag{1.3}$$

where

$$f_0(x) = (p - 1 + b_0 |x|^2)^{-1/(p-1)} \text{ and } b_0 = (p - 1)^2/(4p).$$

Formal arguments for the existence of such a profile were first suggested by Galaktionov and Posashkov [8, 9] in 1985, then, Berger and Kohn [2] gave a numerical confirmation in 1988, and the proof came from Herrero and Velázquez [14] in 1993 and Brimont and Kupiainen [3] in 1994. Later, Merle and Zaag [17] simplified the proof of [3] and proved the stability of the profile f_0 .

The authors in [3] and [17] used a *constructive* proof, based on:

- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

Let us add that other profiles are possible (see [3] and [14]), and that f_0 was proved to be generic by Herrero and Velázquez in [13] and [12] (the proof was given only in one space dimension, and the authors asserted the proof holds also in higher dimensions).

When $\mu \neq 0$ and $q < q_c$, Ebde and Zaag [7] were able to show that the existence of a solution with the same profile f_0 as for the case $\mu = 0$. This is reasonable, since in similarity variables defined below by (2.1), the gradient term comes with an exponentially decreasing term. However, some involved parabolic regularity arguments were needed in [7] to handle the gradient term.

When $\mu \neq 0$ and $q = q_c$, up to our knowledge, there is only one result proving the existence of blow-up solutions for equation (1.1): if $\mu < 0$ and $p - 1$ is small, Souplet, Tayachi and Weissler constructed a selfsimilar blow-up solution in [29]. Let us also mention the numerical result by Nguyen in [34] who finds the same behavior as in (1.3) with almost the same profile as f_0 , in the sense that only the constant b_0 changes into b_μ , continuous in terms of μ (let us also mention the solution by Galaktionov and Vázquez in [10] and [11] in the supercritical case $q = 2 > q_c$ with $\mu > 0$).

In [32] where we consider the critical case $q = q_c$, we initially wanted to prove rigorously the numerical result by Nguyen, but we didn't succeed. We ended instead by finding a *new* type of blow-up behavior, different from (1.3), in the case (1.2), as we state in the following result:

Theorem 1.1. (Blow-up profile for Equation (1.1)) *For any $\varepsilon > 0$, Equation (1.1) has a solution $u(x, t)$ such that u and ∇u blow up in finite time $T > 0$ simultaneously at the origin and only there. Moreover:*

(i) *For all $t \in [0, T)$,*

$$\left\| (T - t)^{\frac{1}{p-1}} u(x, t) - \left(p - 1 + \frac{b|x|^2}{(T - t)|\log(T - t)|^\beta} \right)^{-\frac{1}{p-1}} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{1 + |\log(T - t)|^{\min(\frac{2}{p-1}, \frac{p-3}{2(p-1)}) - \varepsilon}}, \quad (1.4)$$

where

$$\beta = \frac{p+1}{2(p-1)}, \quad b = \frac{1}{2}(p-1)^{\frac{p-2}{p-1}} \left(\frac{(4\pi)^{\frac{N}{2}}(p+1)^2 N}{p \int_{\mathbb{R}^N} |y|^q e^{-|y|^2/4} dy} \right)^{\frac{p+1}{p-1}} \mu^{-\frac{p+1}{p-1}} > 0, \quad (1.5)$$

and a similar estimate holds for ∇u .

(ii) *For all $x \neq 0$, $u(x, t) \rightarrow u^*(x)$ as $t \rightarrow T$ in $C^1(\frac{1}{R} < |x| < R)$ for any $R > 0$, with*

$$u^*(x) \sim \left(\frac{b|x|^2}{[2|\log|x||]^{\frac{p+1}{p-1}}} \right)^{-\frac{1}{p-1}}, \quad \text{as } x \rightarrow 0,$$

and for $|x|$ small, $|\nabla u^*(x)| \leq C \frac{|x|^{-\frac{p+1}{p-1}}}{|\log|x||^\alpha}$, for some $\alpha = \alpha(p, \varepsilon) \in \mathbb{R}$.

Remark 1.1. Note that the solution constructed in the above theorem does not exist in the case of the standard nonlinear heat equation, i.e. when $\mu = 0$ in (1.1).

Indeed, our solution has a profile depending on the reduced variable

$$z = \frac{x}{\sqrt{T-t} |\log(T-t)|^\beta}$$

whereas, we know from the results in [14, 33] that the blow-up profiles in the case $\mu = 0$ depend on the reduced variables

$$z = \frac{x}{\sqrt{T-t} |\log(T-t)|^{\frac{1}{2}}} \text{ or } z = \frac{x}{(T-t)^{\frac{1}{2m}}}, \text{ where } m \geq 2 \text{ is an integer.}$$

As a consequence of our techniques, we also obtained the following stability result in [32]:

Theorem 1.2. *The constructed solution is stable with respect to initial data.*

Let us give an idea of the methods used to prove the results. We construct the blow-up solution with the profile in Theorem 1.1, by following the methods of [3] and [17], though we are far from a simple adaptation, since the gradient term needs genuine new ideas as we explain shortly below. This kind of methods has been applied for various nonlinear evolution equations. For hyperbolic equations, it has been successfully used for the construction of multi-solitons for the semilinear wave equation in one space dimension (see [6]). For parabolic equations, it has been used in [15] and [35] for the complex Ginzburg-Landau equation with no gradient structure. See also the cases of the wave maps in [20], the Schrödinger maps in [16], the critical harmonic heat flow in [21], the two-dimensional Keller-Segel equation in [22] and the nonlinear heat equation involving a subcritical nonlinear gradient term in [7]. Recently, this method has been applied for a non variational parabolic system in [19] and for a logarithmically perturbed nonlinear heat equation in [18].

Unlike in the subcritical case in [7], the gradient term in the critical case induces substantial changes in the blow-up profile as we pointed-out in the comments following Theorem 1.1. Accordingly, its control requires special arguments. So, working in the framework of [17], some crucial modifications are needed. In particular, we have to overcome the following challenges:

- The prescribed profile is not known and not obvious to find. See Section 2 for a formal approach to justify such a profile, and the introduction of the parameter β given by (2.22) below.
- The profile is different from the profile in [17], hence also from all the previous studies in the parabolic case ([17, 7, 18, 19]). Therefore, brand new estimates are needed. See Section 4 below.
- In order to handle the new parameter β in the profile, we introduce a new shrinking set to trap the solution. See Definition 4.2 below. Finding such a set is not trivial, in particular the limitation $p > 3$ is related to the choice of such a set.
- A good understanding of the dynamics of the linearized operator of equation (2.2) below around the new profile is needed, taking into account the new shrinking set.
- Some crucial global and pointwise estimates of the gradient of the solution as well as fine parabolic regularity results are needed.

Then, following [17], the proof is divided in two steps. First, we reduce the problem to a finite dimensional one. Second, we solve the finite dimensional problem and conclude by contradiction, using index theory.

The stability result, Theorem 1.2, is proved similarly as in [17] by interpreting the finite dimensional problem in terms of the blow-up time and the blow-up point.

Thanks to simple change of variables, we obtain similar statements for the following perturbation of the following viscous Hamilton-Jacobi (vHJ) equations:

$$\partial_t u = \Delta u + |\nabla u|^q + \nu |u|^{p-1} u, \quad \text{with } \nu > 0, \quad 3/2 < q < 2, \quad p = \frac{q}{2-q}. \quad (1.6)$$

Corollary 1.1. (Blow-up in the viscous Hamilton-Jacobi (vHJ) equation) *Theorems 1.1 and 1.2 yield stable blow-up solutions in equation (1.6). Moreover, the solution and its gradient blow up simultaneously and only at one point. The blow-up profile is given by (1.4) with appropriate scaling.*

Remark 1.2. Obviously, our result does not hold for the viscous Hamilton-Jacobi equation with $\nu = 0$. An interesting question is to understand the behavior of the constructed solutions, say u_ν , as $\nu \rightarrow \infty$. In our opinion, this is a difficult open question

This note is organized as follows:

- In Section 2, we explain formally how we obtain the profile and the exponent β ;
- In Section 3, we give a formulation of the problem in order to justify the formal argument;
- In Section 4, we give the proof of the existence of the profile assuming some technical results.

For simplicity, we only focus on the case

$$N = 1,$$

and refer the reader to [32] where the high-dimensional case is presented better. We also refer to [32] for the technical details which are omitted here.

2. A FORMAL APPROACH

The aim of this section is to explain formally how we derive the behavior given in Theorem 1.1. In particular, how we obtain the profile φ^0 in (1.4) (see (2.21) for the notation φ^0), the parameter b and the exponent $\beta = 2(p+1)/(p-1)$ in (1.5). We will also explain why our strategy works only for $\mu > 0$, as asserted in (1.2), and not in the case $\mu < 0$ (of course, we never consider the case $\mu = 0$ which corresponds to the well-known semilinear heat equation). For that purpose, we only assume here that

$$\mu \neq 0,$$

and we will explain at the end of this section why we need the positivity assumption on μ (see (2.20) below).

Let us consider an arbitrary $T > 0$ and the self-similar transformation of (1.1)

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T - t). \quad (2.1)$$

It follows that if $u(x, t)$ satisfies (1.1) for all $(x, t) \in \mathbb{R} \times [0, T)$, then $w(y, s)$ satisfies the following equation:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{1}{p-1} w + \mu |\partial_y w|^q + |w|^{p-1} w, \quad (2.2)$$

for all $(y, s) \in \mathbb{R} \times [-\log T, \infty)$. Thus, constructing a solution $u(x, t)$ for the equation (1.1) that blows up at $T < \infty$ like $(T - t)^{-\frac{1}{p-1}}$ reduces to constructing a global solution $w(y, s)$ for equation (2.2) such that

$$0 < \varepsilon \leq \limsup_{s \rightarrow \infty} \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\varepsilon}. \quad (2.3)$$

A first idea to construct a blow-up solution for (1.1), would be to find a stationary solution for (2.2), yielding a self-similar solution for (1.1). It happens that when $\mu < 0$ and p is close to 1, the first author together with Souplet and Weissler were able in [29] to construct such a solution. Now, if $\mu > 0$, we know, still from [29] that it is not possible to construct such a solution in some restrictive class of solutions (see [29, Remark 2.1, p. 666]), of course, apart from the trivial constant solution $w \equiv \kappa$ of (2.2), where

$$\kappa = \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}}. \quad (2.4)$$

2.1. Inner expansion. Following the approach of Bricmont and Kupiainen in [3], we may look for a solution w such that $w \rightarrow \kappa$ as $s \rightarrow \infty$. Writing

$$w = \kappa + \bar{w},$$

we see that $\bar{w} \rightarrow 0$ as $s \rightarrow \infty$ and satisfies the equation:

$$\partial_s \bar{w} = \mathcal{L} \bar{w} + \bar{B}(\bar{w}) + \mu |\nabla \bar{w}|^q, \quad (2.5)$$

where

$$\mathcal{L} = \partial_y^2 - \frac{1}{2} y \partial_y + 1, \quad (2.6)$$

and

$$\bar{B}(\bar{w}) = |\bar{w} + \kappa|^{p-1}(\bar{w} + \kappa) - \kappa^p - p\kappa^{p-1}\bar{w}. \quad (2.7)$$

Note that

$$|\bar{B}(\bar{w}) - \frac{p}{2\kappa} \bar{w}^2| \leq C |\bar{w}^3|,$$

where C is a positive constant.

Let us recall some properties of \mathcal{L} . The operator \mathcal{L} is self-adjoint in $D(\mathcal{L}) \subset L_\rho^2(\mathbb{R})$ where

$$L_\rho^2(\mathbb{R}) = \left\{ f \in L_{loc}^2(\mathbb{R}) \mid \int_{\mathbb{R}} (f(y))^2 \rho(y) dy < \infty \right\}$$

and

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}, \quad y \in \mathbb{R}.$$

The spectrum of \mathcal{L} is explicitly given by

$$\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \mid m \in \mathbb{N} \right\}.$$

It consists only in eigenvalues, which are all simple, and the eigenfunctions are dilations of Hermite polynomials: the eigenvalue $1 - \frac{m}{2}$ corresponds to the following eigenfunction:

$$h_m(y) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}. \quad (2.8)$$

In particular $h_0(y) = 1$, $h_1(y) = y$ and $h_2(y) = y^2 - 2$. Notice that h_m satisfies:

$$\int_{\mathbb{R}} h_n h_m \rho dx = 2^n n! \delta_{nm} \text{ and } \mathcal{L}h_m = \left(1 - \frac{m}{2}\right) h_m.$$

In compliance with the spectral properties of \mathcal{L} , we may look for a solution expanded as follows:

$$\bar{w}(y, s) = \sum_{m \in \mathbb{N}} \bar{w}_m(s) h_m(y).$$

Since h_m , for $m \geq 3$ correspond to negative eigenvalues of \mathcal{L} , assuming \bar{w} even in y , we may consider that

$$\bar{w}(y, s) = \bar{w}_0(s) + \bar{w}_2(s) h_2(y), \quad (2.9)$$

with $\bar{w}_0, \bar{w}_2 \rightarrow 0$ as $s \rightarrow \infty$.

Projecting Equation (2.5), and writing $\mu |\nabla \bar{w}|^q = \mu 2^q |y|^q |\bar{w}_2|^q$, we derive the following ODE system for \bar{w}_0 and \bar{w}_2 :

$$\begin{aligned} \bar{w}_0' &= \bar{w}_0 + \frac{p}{2\kappa} (\bar{w}_0^2 + 8\bar{w}_2^2) + \tilde{c}_0 |\bar{w}_2|^q + O(|\bar{w}_0|^3 + |\bar{w}_2|^3), \\ \bar{w}_2' &= 0 + \frac{p}{\kappa} (\bar{w}_0 \bar{w}_2 + 4\bar{w}_2^2) + \tilde{c}_2 |\bar{w}_2|^q + O(|\bar{w}_0|^3 + |\bar{w}_2|^3), \end{aligned}$$

where

$$\tilde{c}_0 = \mu 2^q \int_{\mathbb{R}} |y|^q \rho \text{ and } \tilde{c}_2 = \frac{\mu 2^q}{8} \int_{\mathbb{R}} |y|^q (|y|^2 - 2) \rho.$$

Note that for this calculation, we need to know the values of

$$\int_{\mathbb{R}} (|y|^2 - 2)^2 \rho(y) dy = 8 \text{ and } \int_{\mathbb{R}} (|y|^2 - 2)^3 \rho(y) dy = 64.$$

Note also that the sign of \tilde{c}_0 and \tilde{c}_2 is the same as for μ . Indeed, obviously $\int_{\mathbb{R}^N} |y|^q \rho(y) dy > 0$, and for $\int_{\mathbb{R}} |y|^q (|y|^2 - 2) \rho(y) dy$, using integration by parts, we write

$$\begin{aligned} \frac{8\tilde{c}_2}{2^q \mu} &= \int_{\mathbb{R}} |y|^q (|y|^2 - 2) \rho(y) dy = \int_{\mathbb{R}} |y|^{q+2} \rho(y) dy - 2 \int_{\mathbb{R}} |y|^q \rho(y) dy \\ &= 2(q+1) \int_{\mathbb{R}} |y|^q \rho(y) dy - 2 \int_{\mathbb{R}} |y|^q \rho(y) dy = 2q \int_{\mathbb{R}} |y|^q \rho(y) dy > 0. \end{aligned} \quad (2.10)$$

From the equation on \bar{w}_2' , we write

$$\bar{w}_2' = \tilde{c}_2 |\bar{w}_2|^q (1 + O(|\bar{w}_2|^{2-q})) + \frac{p}{\kappa} \bar{w}_0 \bar{w}_2 + O(|\bar{w}_0|^3),$$

and assuming that

$$|\bar{w}_0 \bar{w}_2| \ll |\bar{w}_2|^q, \quad |\bar{w}_0|^3 \ll |\bar{w}_2|^q, \quad (2.11)$$

we get that

$$\bar{w}_2' \sim \text{sign}(\mu) |\tilde{c}_2| |\bar{w}_2|^q,$$

with $\text{sign}(\mu) = 1$ if $\mu > 0$ and -1 if $\mu < 0$.

In particular, if $\mu > 0$, then \bar{w}_2 is increasing tending to 0 as $s \rightarrow \infty$ hence $\bar{w}_2 < 0$, while if $\mu < 0$, \bar{w}_2 is decreasing tending to 0 as $s \rightarrow \infty$, hence $\bar{w}_2 > 0$. Then, since $1 < q < 2$, we get

$$\bar{w}_2 \sim -\text{sign}(\mu) \frac{B}{s^{\frac{1}{q-1}}},$$

with

$$B = [(q-1)|\tilde{c}_2|]^{-\frac{1}{q-1}} = \left[2^{q-2} q (q-1) |\mu| \int_{\mathbb{R}} |y|^q \rho \right]^{-\frac{1}{q-1}} \quad (2.12)$$

from (2.10).

From the equation on \bar{w}'_0 , we write

$$\bar{w}'_0 = \bar{w}_0 (1 + O(\bar{w}_0)) + \tilde{c}_0 |\bar{w}_2|^q (1 + O(|\bar{w}_2|^{2-q})),$$

and assuming that

$$|\bar{w}'_0| \ll \bar{w}_0, \quad |\bar{w}'_0| \ll |\bar{w}_2|^q, \quad (2.13)$$

we derive that

$$\bar{w}_0 \sim -\tilde{c}_0 |\bar{w}_2|^q \sim \frac{-\tilde{c}_0 B^q}{s^{\frac{q}{q-1}}} \ll |\bar{w}_2|.$$

Such \bar{w}_0 and \bar{w}_2 are compatible with the hypotheses (2.11) and (2.13).

Therefore, since $w = \kappa + \bar{w}$, it follows from (2.9) that

$$\begin{aligned} w(y, s) &= \kappa + \bar{w}_2(s)(|y|^2 - 2) + o(\bar{w}_2) \\ &= \kappa - \frac{\text{sign}(\mu)}{s^{\frac{1}{q-1}}} B(|y|^2 - 2) + o\left(\frac{1}{s^{\frac{1}{q-1}}}\right) \\ &= \kappa - \text{sign}(\mu) B \frac{|y|^2}{s^{\frac{1}{q-1}}} + 2 \frac{\text{sign}(\mu)}{s^{\frac{1}{q-1}}} B + o\left(\frac{1}{s^{\frac{1}{q-1}}}\right), \end{aligned} \quad (2.14)$$

in $L^2_\rho(\mathbb{R})$, and also uniformly on compact sets by standard parabolic regularity.

2.2. Outer expansion. From (2.14), we see that the variable

$$z = \frac{y}{s^\beta}, \quad \text{with } \beta = \frac{1}{2(q-1)} = \frac{p+1}{2(p-1)},$$

as given in (1.5), is perhaps the relevant variable for blow-up. Unfortunately, (2.14) provides no shape, since it is valid only on compact sets (note that $z \rightarrow 0$ as $s \rightarrow \infty$ in this case). In order to see some shape, we may need to go further in space, to the “outer region”, namely when $z \neq 0$. In view of (2.14), we may try to find an expression of w of the form

$$w(y, s) = \varphi^0(z) + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right), \quad (2.15)$$

for some $\nu > 2\beta$. Plugging this ansatz in equation (2.2), keeping only the main order, we end-up with the following equation on φ^0 :

$$-\frac{1}{2} z [\varphi^0]'(z) - \frac{1}{p-1} \varphi^0(z_0) + [\varphi^0(z)]^p = 0, \quad z = \frac{y}{s^\beta}. \quad (2.16)$$

Recalling that our aim is to find w a solution of (2.2) such that $w \rightarrow \kappa$ as $s \rightarrow \infty$ (in L^2_ρ , hence uniformly on every compact set), we derive from (2.15) (with $y = z = 0$) the natural condition

$$\varphi^0(0) = \kappa.$$

Recalling also that we already adopted radial symmetry for the inner equation, we do the same here. Therefore, integrating equation (2.16), we see that

$$\varphi^0(z) = \left(p-1 + b|z|^2\right)^{-\frac{1}{p-1}}, \quad (2.17)$$

for some $b \in \mathbb{R}$. Recalling also that we want a solution $w \in L^\infty(\mathbb{R})$, (see (2.3)), we see that $b \geq 0$ and for a nontrivial solution, we should have

$$b > 0. \quad (2.18)$$

Thus, we have just obtained from (2.15) that

$$w(y, s) = \left(p - 1 + b|z|^2\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right), \quad \text{with } z = \frac{y}{s^\beta} \text{ and } \nu > 2\beta. \quad (2.19)$$

We should understand this expansion to be valid at least on compact sets in z , that is for $|y| < Rs^\beta$, for any $R > 0$.

2.3. Matching asymptotics. Since (2.19) holds for $|y| < Rs^\beta$, for any $R > 0$, it holds also uniformly on compact sets, leading to the following expansion for y bounded:

$$w(y, s) = \kappa - \frac{\kappa b}{(p-1)^2} \frac{|y|^2}{s^{2\beta}} + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right).$$

Comparing with (2.14), we find the following values for b and a :

$$b = \text{sign}(\mu) \frac{B(p-1)^2}{\kappa} \quad \text{and} \quad a = 2\text{sign}(\mu)B.$$

In particular, from (2.18) we see that

$$\mu > 0. \quad (2.20)$$

In conclusion, using (2.12), we see that we have just derived the following profile for $w(y, s)$:

$$w(y, s) \sim \varphi(y, s)$$

with

$$\varphi(y, s) = \varphi^0\left(\frac{y}{s^\beta}\right) + \frac{a}{s^{2\beta}} := \left(p - 1 + b\frac{|y|^2}{s^{2\beta}}\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}}. \quad (2.21)$$

$$\beta = \frac{p+1}{2(p-1)}, \quad (2.22)$$

$$a = \frac{2b\kappa}{(p-1)^2}, \quad (2.23)$$

$$b = \frac{1}{2}(p-1)^{\frac{p-2}{p-1}} \left(\frac{2\sqrt{\pi}(p+1)^2}{p \int_{\mathbb{R}} |y|^q e^{-|y|^2/4} dy} \right)^{\frac{p+1}{p-1}} \mu^{-(p+1)/(p-1)}, \quad (2.24)$$

3. FORMULATION OF THE PROBLEM

In this section we formulate the problem in order to justify the formal approach given in the previous section. Let w , y and s be as in (2.1). Let us introduce $v(y, s)$ such that

$$w(y, s) = \varphi(y, s) + v(y, s), \quad (3.1)$$

where φ is given by (2.21). If w satisfies the equation (2.2), then v satisfies the following equation:

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(v) + R(y, s), \quad (3.2)$$

where \mathcal{L} is defined by (2.6) and

$$V(y, s) = p \varphi(y, s)^{p-1} - \frac{p}{p-1}, \quad (3.3)$$

$$B(v) = |\varphi + v|^{p-1}(\varphi + v) - \varphi^p - p\varphi^{p-1}v, \quad (3.4)$$

$$R(y, s) = \partial_y^2 \varphi - \frac{1}{2} y \partial_y \varphi - \frac{\varphi}{p-1} + \varphi^p - \frac{\partial \varphi}{\partial s} + \mu |\partial_y \varphi|^q \quad (3.5)$$

and

$$G(v) = \mu|\partial_y \varphi + \partial_y v|^q - \mu|\partial_y \varphi|^q. \quad (3.6)$$

Our aim is to construct initial data $v(s_0)$ such that the equation (3.2) has a solution $v(y, s)$ defined for all $(y, s) \in \mathbb{R} \times [-\log T, \infty)$, and satisfies:

$$\lim_{s \rightarrow \infty} \|v(s)\|_{W^{1,\infty}(\mathbb{R})} = 0. \quad (3.7)$$

From Equation (2.21), one sees that the variable $z = \frac{y}{s^\beta}$ plays a fundamental role. Thus we will consider the dynamics for $|z| > K$ and $|z| < 2K$ separately for some $K > 0$ to be fixed large. Since

$$|B(v)| \leq C|v|^2, \quad \|R(\cdot, s)\|_{L^\infty} \leq \frac{C}{s}, \quad \|G(v)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{s}} \|v\|_{L^\infty(\mathbb{R})}, \quad (3.8)$$

for s large enough (see [32]), it is then reasonable to think that the dynamics of equation (3.2) are influenced by the linear part, namely $\mathcal{L} + V$.

The properties of the operator \mathcal{L} were given in Section 2. In particular, \mathcal{L} is predominant on all the modes, except on the null modes where the terms Vv and $G(v)$ will play a crucial role (see [32]).

As for the potential V , it has two fundamental properties which will strongly influence our strategy:

- (i) we have $V(\cdot, s) \rightarrow 0$ in $L^2_\rho(\mathbb{R})$ when $s \rightarrow \infty$. In practice, the effect of V in the blow-up area ($|y| \leq Cs^\beta$) is regarded as a perturbation of the effect of \mathcal{L} (except on the null mode).
- (ii) outside of the blow-up area, we have the following property: for all $\epsilon > 0$, there exists $C_\epsilon > 0$ and s_ϵ such that

$$\sup_{s \geq s_\epsilon, \frac{|y|}{s^\beta} \geq C_\epsilon} \left| V(y, s) - \left(-\frac{p}{p-1}\right) \right| \leq \epsilon,$$

with $-\frac{p}{p-1} < -1$. As 1 is the largest eigenvalue of the operator \mathcal{L} , outside the blow-up area we can consider that the operator $\mathcal{L} + V$ is an operator with negative eigenvalues, hence, easily controlled.

Considering the fact that the behavior of V is not the same inside and outside the blow-up area, we decompose v as follows. Let us consider a non-increasing cut-off function $\chi_0 \in C^\infty([0, \infty), [0, 1])$ such that $\text{supp}(\chi_0) \subset [0, 2]$ and $\chi_0 \equiv 1$ in $[0, 1]$, and introduce

$$\chi(y, s) = \chi_0\left(\frac{|y|}{K s^\beta}\right) \quad (3.9)$$

with K is some large enough constant so that various estimates in the proof hold. Then, we write

$$v(y, s) = v_b(y, s) + v_e(y, s), \quad (3.10)$$

with

$$v_b(y, s) = v(y, s)\chi(y, s) \text{ and } v_e(y, s) = v(y, s)(1 - \chi(y, s)). \quad (3.11)$$

We remark that

$$\text{supp } v_b(s) \subset B(0, 2Ks^\beta), \quad \text{supp } v_e(s) \subset \mathbb{R} \setminus B(0, Ks^\beta).$$

As for v_b , we will decompose it according to the sign of the eigenvalues of \mathcal{L} , by writing

$$v_b(y, s) = \sum_{m=0}^2 v_m(s) h_m(y) + v_-(y, s), \quad (3.12)$$

where for $0 \leq m \leq 2$, $v_m = P_m(v_b)$ and $v_-(y, s) = P_-(v_b)$, with P_m the L^2_ρ projector on h_m , the eigenfunction corresponding to $\lambda = 1 - \frac{m}{2} \geq 0$, and P_- the projector on $\{h_i, | i \geq 3\}$, the negative subspace of the operator \mathcal{L} (as announced in the beginning of the section, hereafter, we assume that $N = 1$ for simplicity).

Thus, we can decompose v in five components as follows:

$$v(y, s) = \sum_{m=0}^2 v_m(s) h_m(y) + v_-(y, s) + v_e(y, s). \quad (3.13)$$

Here and throughout the paper, we call v_- the negative mode of v , v_2 the null mode of v , and the subspace spanned by $\{h_m | m \geq 3\}$ will be referred to as the negative subspace.

4. THE EXISTENCE PROOF WITHOUT TECHNICAL DETAILS

In this section, we prove the existence of a solution v of (3.2) such that

$$\lim_{s \rightarrow \infty} \|v(s)\|_{W^{1,\infty}(\mathbb{R})} = 0. \quad (4.1)$$

This is in fact the main step towards the proof of Theorem 1.1. Here, we only give the arguments of the proof, and for the technical details, we refer the interested reader to our paper [32]. For the remaining steps of the proof of Theorem 1.1 and also for the proof of Theorem 1.2, we refer to [32].

Since $p > 3$, we see that, by definition of β given by (2.22), $\beta \in (\frac{1}{2}, 1)$. Our construction is build on a careful choice of the initial data for v at a time s_0 . We will choose it in the following form:

Definition 4.1. Choice of the initial data) *Let us define, for $A \geq 1$, $s_0 = -\log T > 1$ and $d_0, d_1 \in \mathbb{R}$, the function*

$$\psi_{s_0, d_0, d_1}(y) = \frac{A}{s_0^{2\beta+1}} \left(d_0 h_0(y) + d_1 h_1(y) \right) \chi(2y, s_0), \quad (4.2)$$

where h_i , $i = 0, 1$ are defined by (2.8) and χ is defined by (3.9).

The solution of equation (3.2) will be denoted by v_{s_0, d_0, d_1} or v when there is no ambiguity. We will show that if A is fixed large enough, then, s_0 is fixed large enough depending on A , we can fix the parameters $(d_0, d_1) \in [-2, 2]^2$, so that the solution $v_{s_0, d_0, d_1}(s) \rightarrow 0$ as $s \rightarrow \infty$ in $W^{1,\infty}(\mathbb{R})$, that is, (4.1) holds. Owing to the decomposition given in (4.2), it is enough to control the solution in a shrinking set defined as follows:

Definition 4.2. (A set shrinking to zero) *Let γ be any real number such that*

$$3\beta < \gamma < \min(5\beta - 1, 2\beta + 1). \quad (4.3)$$

For all $A \geq 1$ and $s \geq 1$, we define $\vartheta_A(s)$ as the set of all functions $r \in L^\infty(\mathbb{R})$ such that

$$\|r_e\|_{L^\infty(\mathbb{R})} \leq \frac{A^2}{s^{\gamma-3\beta}}, \quad \left\| \frac{r_-(y)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq \frac{A}{s^\gamma},$$

$$|r_0|, |r_1| \leq \frac{A}{s^{2\beta+1}}, \quad |r_2| \leq \frac{\sqrt{A}}{s^{4\beta-1}},$$

where r_- , r_e and r_m are defined in (3.13).

Remark 4.1. Since $p > 3$, it follows that $\frac{1}{2} < \beta < 1$, in particular the range for γ in (4.3) is not empty. Of course, the set $\vartheta_A(s)$ depends also on the choice of γ satisfying (4.3). However, while A will be chosen large enough so that various estimates hold, γ will be fixed once for all throughout the proof.

Since $A \geq 1$, then the sets $\vartheta_A(s)$ are increasing (for fixed s) with respect to A in the sense of inclusion. We also show the following property of elements of $\vartheta_A(s)$:

For all $A \geq 1$, there exists $s_{01}(A) \geq 1$ such that, for all $s \geq s_{01}$ and $r \in \vartheta_A(s)$, we have

$$\|r\|_{L^\infty(\mathbb{R})} \leq C \frac{A^2}{s^{\gamma-3\beta}}, \quad (4.4)$$

where C is a positive constant (see [32]).

By (4.4), if a solution v stays in $\vartheta_A(s)$ for $s \geq s_0$, then it converges to 0 in $L^\infty(\mathbb{R})$ (the convergence of the gradient will follow from parabolic regularity). Reasonably, our aim is then reduced to prove the following proposition:

Proposition 4.1. (Existence of solutions trapped in $\vartheta_A(s)$) *There exists $A_2 \geq 1$ such that for $A \geq A_2$ there exists $s_{02}(A)$ such that for all $s_0 \geq s_{02}(A)$, there exists (d_0, d_1) such that if v is the solution of (3.13) with initial data at s_0 , given by (4.2), then $v(s) \in \vartheta_A(s)$, for all $s \geq s_0$.*

This proposition gives the stronger convergence to 0 in $L^\infty(\mathbb{R})$ thanks to (4.4), and the convergence in $W^{1,\infty}(\mathbb{R})$ will follow from an involved parabolic regularity argument as explained in [32].

Let us first make sure that we can choose the initial data such that it starts in $\vartheta_A(s_0)$. In other words, we will define a set where we will at the end select the good parameter (d_0, d_1) that will give the conclusion of Proposition 4.1. More precisely, we have the following result:

Proposition 4.2. (Properties of initial data) *For each $A \geq 1$, there exists $s_{03}(A) > 1$ such that for all $s_0 \geq s_{03}$, there exists a rectangle*

$$\mathcal{D}_{s_0} \subset [-2, 2]^2$$

such that the mapping

$$\begin{aligned} \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \\ (d_0, d_1) &\mapsto (\psi_0, \psi_1). \end{aligned}$$

(where ψ stands for ψ_{s_0, d_0, d_1}) is linear, one to one from \mathcal{D}_{s_0} onto $[-\frac{A}{s_0^{2\beta+1}}, \frac{A}{s_0^{2\beta+1}}]^2$ and maps $\partial\mathcal{D}_{s_0}$ into $\partial\left([-\frac{A}{s_0^{2\beta+1}}, \frac{A}{s_0^{2\beta+1}}]^2\right)$. Moreover, it has degree one on the boundary.

Proof. See [32].

Proof of Proposition 4.1. Let us consider $A \geq 1$, $s_0 \geq s_{03}$, $(d_0, d_1) \in \mathcal{D}_{s_0}$, where s_{03} is given by Proposition 4.2. From the existence theory (which follows from the Cauchy problem for equation (1.1) in $W^{1,\infty}(\mathbb{R})$) mentioned in the introduction),

starting in $\vartheta_A(s_0)$ which is in $\vartheta_{A+1}(s_0)$, the solution stays in $\vartheta_A(s)$ until some maximal time $s_* = s_*(d_0, d_1)$. If $s_*(d_0, d_1) = \infty$ for some $(d_0, d_1) \in \mathcal{D}_{s_0}$, then the proof is complete. Otherwise, we argue by contradiction and suppose that $s_*(d_0, d_1) < \infty$ for any $(d_0, d_1) \in \mathcal{D}_{s_0}$. By continuity and the definition of s_* , the solution at the point s_* , is on the boundary of $\vartheta_A(s_*)$. Then, by definition of $\vartheta_A(s_*)$, one at least of the inequalities in that definition is an equality. Owing to the following proposition, this can happen only for the first two components. Precisely, we have the following result:

Proposition 4.3. (Control of $v(s)$ by $(v_0(s), v_1(s))$ in $\vartheta_A(s)$) *There exists $A_4 \geq 1$ such that for each $A \geq A_4$, there exists $s_{04}(A) \in \mathbb{R}$ such that for all $s_0 \geq s_{04}(A)$, the following holds:*

If v is a solution of (3.2) with initial data at $s = s_0$ given by (4.2) with $(d_0, d_1) \in \mathcal{D}_{s_0}$, and $v(s) \in \vartheta_A(s)$ for all $s \in [s_0, s_1]$, with $v(s_1) \in \partial\vartheta_A(s_1)$ for some $s_1 \geq s_0$, then:

(i) *(Reduction to a finite dimensional problem) We have:*

$$(v_0(s_1), v_1(s_1)) \in \partial \left(\left[-\frac{A}{s_1^{2\beta+1}}, \frac{A}{s_1^{2\beta+1}} \right]^2 \right).$$

(ii) *(Transverse crossing) There exist $m \in \{0, 1\}$ and $\omega \in \{-1, 1\}$ such that*

$$\omega v_m(s_1) = \frac{A}{s_1^{2\beta+1}} \text{ and } \omega v'_m(s_1) > 0.$$

Assume the result of the previous proposition, for which the proof is given in [32], and continue the proof of Proposition 4.1. Let $A \geq A_4$ and $s_0 \geq s_{04}(A)$. It follows from Proposition 4.3, part (i), that $(v_0(s_*), v_1(s_*)) \in \partial \left(\left[-\frac{A}{s_*^{2\beta+1}}, \frac{A}{s_*^{2\beta+1}} \right]^2 \right)$, and the following function

$$\begin{aligned} \Phi : \mathcal{D}_{s_0} &\rightarrow \partial([-1, 1]^2) \\ (d_0, d_1) &\mapsto \frac{s_*^{2\beta+1}}{A} (v_0, v_1)_{(d_0, d_1)}(s_*), \text{ with } s_* = s_*(d_0, d_1), \end{aligned}$$

is well defined. Then, it follows from Proposition 4.3, part (ii) that Φ is continuous. On the other hand, using Proposition 4.2, parts (i) and (ii) together with the fact that $v(s_0) = \psi_{s_0, d_0, d_1}$, we see that when (d_0, d_1) is on the boundary of the rectangle \mathcal{D}_{s_0} , we have strict inequalities for the other components. Applying the transverse crossing property given in Proposition 4.3, part (ii), we see that $v(s)$ leaves $\vartheta_A(s)$ at $s = s_0$, hence $s_*(d_0, d_1) = s_0$. Using Proposition 4.2, part (i), we see that the restriction of Φ to the boundary is of degree 1. A contradiction then follows from the index theory. Thus, there exists a value $(d_0, d_1) \in \mathcal{D}_{s_0}$ such that for all $s \geq s_0$, $v_{s_0, d_0, d_1}(s) \in \vartheta_A(s)$. This concludes the proof of Proposition 4.1.

Completion of the proof of (4.1). By Proposition 4.1 and (4.4), it remains only to show that $\|\nabla v(s)\|_{L^\infty(\mathbb{R})} \rightarrow 0$ as $s \rightarrow \infty$. This in fact follows from a very involved parabolic regularity argument given in [32], which implies that there exists s_{05} such that for $s \geq s_{05}$,

$$\|v(s)\|_{W^{1,\infty}(\mathbb{R})} \leq \frac{C(A)}{s^{\gamma-3\beta}},$$

hence, by (4.3), (4.1) follows by taking $s_{02} \geq \max(s_{01}, s_{03}, s_{04}, s_{05})$.

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